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The general metric for N-dimensional spherically symmetric and conformally flat spacetimes is given, and all the homogeneous and isotropic solutions for a perfect fluid with the equation of state $p = \alpha\rho$ are found. These solutions are then used to model the gravitational collapse of a compact ball. It is found that when the collapse has continuous self-similarity, the formation of black holes always starts with zero mass, and when the collapse has no such a symmetry, the formation of black holes always starts with a mass gap.

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I. INTRODUCTION

Recently, we have studied the gravitational collapse of perfect fluid in four-dimensional spacetimes [1,2], and found that when solutions have continuous self-similarity, the formation of black holes *always* starts with zero-mass, while when solutions have no such a symmetry it starts with a mass gap. The solutions with zero masses actually represent naked singularities. Thus, if the Cosmic Censorship Conjecture is correct [3], it seems that Nature prohibits the existence of solutions of the Einstein field equations with self-similarity.

Quite recently, there has been renewed interest in studying higher dimensional spacetimes from the point of view of both Cosmology [4] and gravitational collapse [5]. In particular, it was found that the exponent γ, appearing in the mass scaling form of black holes, depends on the dimensions of the spacetimes [6].

In this paper, we shall generalize our previous studies to the case of perfect fluid in N-dimensional spherical spacetimes and will show that our previous results for gravitational collapse, obtained in four-dimensional spacetimes [1,2], are also valid in N-dimensional spacetimes. The solutions to be presented below can be also considered as representing cosmological models. However, in this paper we shall not consider this possibility and leave it to another occasion. The rest of the paper is organized as follows: In Sec.II we shall derive the general form of metric for conformally flat spherical spacetimes with N-dimensions. As an application, all the homogeneous and isotropic Friedmann-Robertson-Walker-like solutions for a perfect fluid with the equation of state $p = \alpha\rho$ are found for spacetimes with any dimensions. The solutions can be classified into three different families, flat, open, and close, depending on the curvature of the $(N-1)$-dimensional spatial part of the metric, as that in four-dimensional case. In Sec. III, two families of these solutions are studied in the context of gravitational collapse, one has continuous self-similarity, and the other has neither continuous self-similarity nor discrete self-similarity. In order to study gravitational collapse in N-dimensional spacetimes, in this section a definition for the mass function is first given, whereby the location of the apparent horizons can be read off directly. This definition gives the correct results for the static case [7] and reduces to the four-dimensional one when $N = 4$ [8]. The paper is ended with Sec. IV, where our main conclusions are derived.

It should be noted that multidimensional spacetimes where all dimensions are in the equal foot, like the ones to be considered here, are not so realistic, as we are living in an effectively 4-dimensional spacetime, so in principle one might expect that by dimensional reduction the multidimensional spacetimes should reduce to our 4-dimensional world. To have such reduction possible, the dimensions should have different weights in any realistic model. In this sense, the models considered in this paper are very ideal.

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II. THE GENERAL HOMOGENEOUS AND ISOTROPIC SOLUTIONS IN N-DIMENSIONAL SPACETIMES

To start with, let us consider the general metric for N-dimensional spacetimes with spherical symmetry [9],

$$ds^2 = G(t,r)dt^2 - K(t,r)\ \left(dr^2 + r^2 d\Omega^2_{N-2}\right), \tag{1}$$

where $\{x^\mu\} \equiv \{t, r, \theta^2, \theta^3, ..., \theta^{N-1}\}$ $(\mu = 0, 1, 2, ..., N-1)$ are the usual N-dimensional spherical coordinates, $d\Omega^2_{N-2}$ is the line element on the unit (N-2)-sphere, given by

$$\begin{aligned} d\Omega^2_{N-2} &= \left(d\theta^2\right)^2 + \sin^2(\theta^2)\left(d\theta^3\right)^2 + \sin^2(\theta^2)\sin^2(\theta^3)\left(d\theta^4\right)^2 \\ &\quad +... + \sin^2(\theta^2)\sin^2(\theta^3)...\sin^2(\theta^{N-2})\left(d\theta^{N-1}\right)^2 \\ &= \sum_{i=2}^{N-1}\left[\prod_{j=2}^{i-1}\sin^2(\theta^j)\right]\left(d\theta^i\right)^2. \end{aligned} \tag{2}$$

Then, it can be shown that the conformally flat condition $C_{\mu\nu\lambda\delta} = 0$, where $C_{\mu\nu\lambda\delta}$ denotes the Weyl tensor, reduces to a single equation

$$D_{,rr} - \frac{D_{,r}}{r} = 0, \tag{3}$$

where $D \equiv (G/K)^{1/2}$, and $(\ )_{,r} \equiv \partial(\ )/\partial r$, etc. The general solution of the above equation is given by

$$D(t,r) = f_1(t) + f_2(t) r^2, \tag{4}$$

where f_1 and f_2 are two arbitrary functions of t only. Using the freedom of coordinate transformations, it can be shown that there are essentially only two different cases, one is $f_1(t) = 1,\ f_2(t) = 0$, and the other is $f_1(t) \neq 0,\ f_2(t) = 1$. Thus, it is concluded that *the general conformally flat N-dimensional metric with spherically symmetry* takes the form

$$ds^2 = G(t,r)\left[dt^2 - h^2(t,r)\ \left(dr^2 + r^2 d\Omega^2_{N-2}\right)\right], \tag{5}$$

where

$$h(t,r) = \left\{\begin{cases} 1, \\ \left[f_1(t) + r^2\right]^{-1}, cr \end{cases}\right. \tag{6}$$

with $f_1(t) \neq 0$. In the following we shall refer solutions with $h(t,r) = 1$ as Type A solutions, and solutions with $h(t,r) = [f_1(t) + r^2]^{-1}$ as Type B solutions.

When $f_1(t) = Const.$, say, f_1, we can introduce a new radial coordinate $\bar{r}$ via the relation

$$\bar{r} = \frac{r}{f_1 + r^2}, \tag{7}$$

then the metric (2) becomes

$$ds^2 = G(t,r)\left(dt^2 - \frac{d\bar{r}^2}{1 - 4f_1\bar{r}^2} - \bar{r}^2 d\Omega^2_{N-2}\right), \quad (f_1(t) = Const.). \tag{8}$$

If we further set $G(t,r) = G(t)$, the above metric becomes the Friedmann-Robertson-Walker (FRW) metric but in N-dimensional spacetimes. Solving the corresponding Einstein field equations for a perfect fluid with the equation of state $p = \alpha\rho$, we find two classes of solutions, where ρ denotes the energy density of the fluid, p the pressure, and α is an arbitrary constant. The details of the derivation of these solutions were given in [10], so in the following we should only present the final form of the solutions.

Type A solutions. In this case, the general solutions are given by

$$h(t,r) = 1, \quad G(t,r) = (1 - Pt)^{2\xi}, \tag{9}$$

where P is a constant, which characterizes the strength of the spacetime curvature. In particular, when $P = 0$, the spacetime becomes Minkowski. The constant ξ is a function of α and the spacetime dimension N, given by

$$\xi \equiv \frac{2}{(N-3)+\alpha(N-1)}. \tag{10}$$

The corresponding energy density and velocity of the fluid are given, respectively, by

$$\begin{aligned} p &= \alpha\rho = 3\kappa^{-1}(N-1)\alpha\xi^2 P^2 (1-Pt)^{-2(\xi+1)}, \\ u_0 &= (1-Pt)^{\xi}, \qquad u_i = 0,\ (i = 1, 2, ..., N-1). \end{aligned} \tag{11}$$

This class of solutions belongs to the ones studied by Ivashchuk and Melnikov in the context of higher dimensional Cosmology [11]. As a matter of fact, setting $n = 0$ and $N_0 = N - 1$ in their general solutions, we shall obtain the above ones. When $\alpha = 0, (N-1)^{-1}$, the above solutions reduce, respectively, to the one for a dust and radiation fluid, which were also studied recently by Chatterjee and Bhui [12]. The $\alpha = 0$ case was studied in the context of gravitational collapse, too [13]. It can be shown that the curvature of the (N-1)-dimensional spatial part of the metric $t = Const.$ in this case is zero, and the spacetimes correspond to the spatially flat FRW model.

Type B solutions. In this case, the general solutions are given by,

$$h(t,r) = \frac{1}{f_1 + r^2}, \quad G(t) = \left[A\cosh(\omega t) + B\sinh(\omega t)\right]^{2\xi}, \tag{12}$$

where $\omega \equiv 2\sqrt{-f_1}/\xi$, A and B are integration constants, and ξ is defined by Eq.(10). The energy density and velocity of the fluid now are given by

$$\begin{aligned} p &= \alpha\rho = 12\kappa^{-1}N(N-1)\alpha f_1 (A^2 - B^2)\left[A\cosh(\omega t) + B\sinh(\omega t)\right]^{-2(1+\xi)}, \\ u_0 &= \left[A\cosh(\omega t) + B\sinh(\omega t)\right]^{\xi}, \quad u_i = 0,\ (i = 1, 2, ...N-1). \end{aligned} \tag{13}$$

It can be shown that the curvature of the (N-1)-dimensional spatial part of the metric $t = Const.$ in the present case is different from zero. In fact, when $f_1 > 0$, the curvature is positive, and the spacetime is spatially closed, and when $f_1 < 0$, the curvature is negative, and the spacetime is spatially open. The particular solution with $\alpha = -(N-3)/(N-1)$ was found in [14], given by Eqs.(49) and (50), but its physical studies were excluded. As far as we know, the rest of the solutions are new.

It should be noted that the above solutions are valid for any constant α. However, in the rest of the paper we shall consider only the case where $-(N-2)^{-1} \le \alpha \le 1$, so that the three energy conditions, weak, dominant, and strong, are satisfied [15]. When $N = 4$, these solutions reduce to the FRW solutions, which have been studied in the context of gravitational collapse in [1,2]. Therefore, in the following we shall assume that $N \neq 4$.

III. GRAVITATIONAL COLLAPSE OF PERFECT FLUID IN N-DIMENSIONAL SPACETIMES

To study the above solutions in the context of gravitational collapse, we need first to define the local mass function. Recently, Chatterjee and Bhui gave a definition, in generalizing the Cahill and Macvittie mass function in four-dimensional spacetimes [16] to N-dimensional spacetimes [17]. However, in this paper we shall use the following definition for the mass function,

$$1 - \frac{2m(t,r)}{B_N r_{ph}^{N-3}} = -g^{\mu\nu} r_{ph,\mu} r_{ph,\nu}, \tag{14}$$

where

$$B_N = \frac{\kappa\Gamma\left(\frac{N-1}{2}\right)}{2(N-2)\pi^{(N-1)/2}}, \tag{15}$$

with Γ denoting the gamma function, and r_{ph} the geometric radius of the (N-2)-unit sphere. It can be shown that this definition in general is different from that given by Chatterjee and Bhui [17], and reduces to the one usually used in four-dimensional spacetimes when $N = 4$ [8], and yields the correct mass of black holes in N-dimensions for the static spherical spacetimes [7].

In the study of gravitational collapse, another important conception is the apparent horizon, the formation of which indicates the formation of black holes. The apparent horizon in the present case is defined as the outmost boundary of the trapped $(N-2)$-spheres [15]. The location of the trapped $(N-2)$-spheres is the place where the outward normal of the surface, $r_{ph} = Const.$, is null, i.e.,

$$g^{\mu\nu} r_{ph\,,\mu} r_{ph\,,\nu} = 0. \tag{16}$$

Then, the mass function on the apparent horizon is given by

$$M_{AH} = \left. \frac{B_N}{2} r_{ph}^{N-3} \right|_{r=r_{AH}}, \tag{17}$$

where $r = r_{AH}$ is a solution of Eq.(16), which corresponds to the outmost trapped surface. In gravitational collapse M_{AH} is usually taken as the mass of black holes [18]. With the above definition for the mass function, let us study the main properties of the above two types of solutions separately.

A. Type A solutions

The mass function defined by Eq.(14) in this case takes the form

$$m(t, r) = \frac{B_N}{2} \frac{\xi^2 P^2 r^{N-1}}{(1 - Pt)^{2-\xi(N-3)}}, \tag{18}$$

while Eq.(16) has the solution

$$r_{AH} = \frac{1}{\xi} \frac{|1 - Pt|}{|P|}, \tag{19}$$

which represents the location of the apparent horizon of the solutions. When $\xi = 1$, the apparent horizon represents a null surface in the (t, r)-plane, and when $0 \leq \xi < 1$, the apparent horizon is spacelike, and when $\xi = 1$, it is null, while when $\xi > 1$, it is timelike. The spacetime is singular when $t = 1/P$. This can be seen, for example, from the Kretschmann scalar, which now is given by

$$\mathcal{R} \equiv R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 6\xi^2 P^4 \left\{ N - 1 + \xi^2 \left[N - 2 + \sum_{A=1}^{N-3} (N - 2 - A) \right] \right\} (1 - Pt)^{-4(1+\xi)}. \tag{20}$$

When $P > 0$, it can be shown that the singularity always hides behind the apparent horizon, and when $P < 0$, the singularity is naked. In the latter case, the solutions can be considered as representing cosmological models, while in the former the solutions as representing the formation of black holes due to the gravitational collapse of the perfect fluid. Substituting Eq.(19) into Eq.(17) we find that, as $t \to +\infty$, the mass of the black hole becomes infinitely large. To remend this shortage, one may follow [19,1,2] to cut the spacetime along a timelike hypersurface, say, $r = r_0(t)$, and then join the part with $r < r_0(t)$ with an asymptotically flat N-dimensional spacetimes. From Eqs.(11) we can see that the fluid is comoving with the coordinates. Thus, the timelike hypersurface now can be chosen as $r = r_0 = Const.$ Then, it can be seen that at the moment $t_c = -(P\xi r_0 - 1)/P$, the whole ball collapses inside the apparent horizon, so the contribution of the collapsing ball to the total mass of such a formed black hole is given by

$$M_{BH}^F = m(t_c, r_0) = \frac{B_N}{2} \xi^{\xi(N-3)} r_0^{(N-3)(1+\xi)} P^{\xi(N-3)}. \tag{21}$$

From the above expression we can see that the mass is proportional to P, the parameter that characterizes the strength of the initial data of the collapsing ball. Thus, when the initial data is very weak ($P \to 0$), the mass of the formed black hole is very small ($M_{BH} \to 0$). In principle, by properly tuning the parameter P we can make it as small as wanted.

It is interesting to note that this class of solutions admits a homothetic Killing vector,

$$\zeta_0 = -\frac{1 - Pt}{(1 + \zeta)P}, \quad \zeta_1 = \frac{r}{1 + \zeta}, \quad \zeta_i = 0, \ (i = 2, 3, ..., N - 1), \tag{22}$$

which satisfies the conformal Killing equation,

$$\zeta_{\mu;\nu} + \zeta_{\nu;\mu} = 2g_{\mu\nu}. \tag{23}$$

Introducing two new coordinates via the relations,

$$\tilde{t} = \frac{(1-Pt)^{\xi+1}}{(1+\xi)P}, \quad \tilde{r} = r^{1+\xi}, \tag{24}$$

we find that the metric can be written in an explicit self-similar form,

$$ds^2 = d\tilde{t}^2 - \left[(\xi+1)^{-1/\xi} Px\right]^{\frac{2\xi}{\xi+1}} d\tilde{r}^2 - [(\xi+1)Px]^{\frac{2\xi}{\xi+1}} \tilde{r}^2 d\Omega^2_{N-2}, \tag{25}$$

where $x \equiv \tilde{t}/\tilde{r}$ is the self-similar variable.

It is well-known that an irrotational "stiff" fluid ($\alpha = 1$) in four-dimensional spacetimes is energetically equal to a massless scalar field [20]. It can be shown that this also the case for N-dimensional spacetimes. In particular, for the above solutions with $\alpha = 1$, the corresponding massless scalar field ϕ is given by

$$\phi = \pm \left[\frac{N-1}{\kappa(N-2)}\right]^{1/2} \ln(1-Pt) + \phi_0, \tag{26}$$

where ϕ_0 is a constant.

B. Type B solutions

The solutions in this case are given by Eq.(12). When $f_1 > 0$, the spacetime is close, and to have the metric be real the constant B has to be imaginary. The spacetimes are singular when,

$$t|_{f_1>0} = \frac{1}{|\omega|}\arctan\left(\frac{|B|}{A}\right) + 2n\pi, \tag{27}$$

where n is an integer. When $f_1 < 0$, the spacetime is singular only when

$$t|_{f_1<0} = \frac{1}{\omega}\text{arctanh}\left(\frac{B}{A}\right). \tag{28}$$

Therefore, in the following we shall consider only the case where $f_1 < 0$. In this case, to have the energy density of the fluid be non-negative, we need to impose the condition $B^2 \geq A^2$. Then, the metric coefficient $G(t)$ can be written as

$$G = \left(B^2 - A^2\right)^{\xi} \sinh^{2\xi}[\omega(t_0 - \epsilon t)], \tag{29}$$

where $\epsilon = sign(B)$, and t_0 is defined as

$$\sinh(\omega t_0) = \frac{A}{(B^2 - A^2)^{1/2}}.$$

Clearly, the conformal factor $(B^2 - A^2)^{\xi}$ does not play any significant role, without loss of generality, in the following we shall set it to be one. If we further introduce a new radial coordinate via the relation,

$$\bar{r} = -\int h(t,r)dr = \frac{1}{a}\ln\left|\frac{a+r}{a-r}\right|, \tag{30}$$

where $a \equiv (-f_1)^{1/2}$, the corresponding metric takes the form,

$$ds^2 = \sinh^{2\xi}\left[\frac{2}{\xi}(t_0 - \epsilon t)\right]\left\{dt^2 - dr^2 - \frac{\sinh^2(2r)}{4} d\Omega^2_{N-2}\right\}. \tag{31}$$

Note that in writing the above expression, we had set, without loss of generality, $a = 1$, and dropped the bars from r. From this metric, it can be shown that it is not self-similar. In the following we shall use this form of metric for the study of the Type B solutions. The corresponding mass function and Kretschmann scalar are given, respectively, by

$$\begin{aligned} m(r,t) &= \frac{B_N}{2^{N-2}} \sinh^{N-1}(2r) \sinh^{\xi(N-3)-2}\left[2\xi^{-1}(t_0 - \epsilon t)\right], \\ \mathcal{R} &= \frac{96}{\xi^2}\left\{(N-1) + \xi^2\left[N - 2 + \sum_{A=1}^{N-3}(N-2-A)\right]\right\} \times \sinh^{-4(\xi+1)}\left[2\xi^{-1}(t_0 - \epsilon t)\right], \end{aligned} \tag{32}$$

while the apparent horizon is located at

$$r = r_{AH} \equiv \xi^{-1}(t_0 - \epsilon t). \tag{33}$$

From the above equations, we can see that the solutions are singular on the hypersurface $t = \epsilon t_0$. When $\epsilon = +1$, the singularity is hidden behind the apparent horizon, and the solutions represent the formation of black holes from the gravitational collapse of the fluid. When $\epsilon = -1$, the singularity is naked, the solutions can be considered as representing cosmological models or white holes. As in the type A case, the mass of such formed black holes also diverges at the limit $t \to +\infty$. Thus, to have finite masses of black holes, we may also make a "surgery" to the spacetimes. Since the fluid is comoving with the coordinates, too. Without loss of generality, we may also choose the boudary as $r = r_0$. Then, it can be shown that the contribution of the collapsing ball to the total mass of black hole now is given by

$$M^F_{BH} \equiv m_{AH}(\tau_{AH}, r_0) = \frac{B_N}{2^{N-2}} \sinh^{(N-3)(\xi+1)}(2r_0). \tag{34}$$

From the above expression we can see that for any given non-zero r_0, M^F_{BH} is always finite and non-zero. Thus, in the present case black holes start to form with a mass gap.

Finally we note that, similar to the last case, the solution with $\alpha = 1$ also corresponds to a massless scalar field with the scalar field being given by

$$\phi = \pm \left[\frac{N-1}{\kappa(N-2)}\right]^{1/2} \ln\left\{\tanh[(N-2)(t_0 - \epsilon t)]\right\} + \phi_0, \; (\alpha = 1). \tag{35}$$

IV. CONCLUDING REMARKS

The general form of metric for N-dimensional spherically symmetric and conformally flat spacetimes was found. As an application of it, all the Friedmann-Robertson-Walker-like solutions for a perfect fluid with an equation of state $p = \alpha\rho$ were given. These solutions were then used to model the gravitational collapse of a compact ball. It was shown that when the collapse has continuous self-similarity, the formation of black holes always starts with zero mass, and when the collapse has no such a symmetry, the formation of black holes always starts with a finite non-zero mass. This is the same as that obtained in the study of the problem in four-dimensional spacetimes [1,2]. Thus, they provide further evidences to support the speculation that *the formation of black holes always starts with zero-mass for the collapse with self-similarities.*

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